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An Erdős-Ko-Rado Theorem For Signed Sets

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Abstract—A *signed r-set* on $[n] = \{1, \dots, n\}$ is a pair (A, f), where $A \subset [n]$ is an r-set and f is a function from A to $\{-1, 1\}$. A family $\mathcal{A}$ of signed r-sets is *intersecting* if for any (A, f), $(B, g) \in \mathcal{A}$ there exists $x \in A \cap B$ such that $f(x) = g(x)$. In this note, we prove that if $\mathcal{A}$ is an intersecting family of signed r-sets on $[n]$, then $|\mathcal{A}| \le 2^{r-1}\binom{n-1}{r-1}$.

We also present an application of this result to a diameter problem in the grid.

Keywords—Extremal problems, Intersecting families, Discrete isoperimetric inequalities.

1. AN ERDŐS-KO-RADO THEOREM FOR SIGNED SETS

A *signed r-set* on $[n] = \{1, \dots, n\}$ is a pair (A, f), where $A \subset [n]$ is an r-set and f is a function from A to $\{-1, 1\}$. A family $\mathcal{A}$ of signed r-sets is *intersecting* if for any $(A, f), (B, g) \in \mathcal{A}$ there exists $x \in A \cap B$ such that $f(x) = g(x)$.

Our aim is to prove that if $\mathcal{A}$ is an intersecting family of signed r-sets on $[n]$ then $|\mathcal{A}| \le 2^{r-1}\binom{n-1}{r-1}$. Note that this inequality is best possible: writing $S_{n,r}$ for the collection of all signed r-sets on $[n]$, for any $x_0 \in [n]$ and $\epsilon_0 \in \{-1, 1\}$, the family $\{(A, f) \in S_{n,r} : x_0 \in A,\ f(x_0) = \epsilon_0\}$ is an intersecting family of signed r-sets of size $2^{r-1}\binom{n-1}{r-1}$.

Note also that if $\mathcal{A}$ is an intersecting family of signed r-sets on $[n]$ then, for each r-set A, there can be at most 2^{r-1} functions $f : A \to \{-1, 1\}$ such that $(A, f) \in \mathcal{A}$. Furthermore, the system of underlying r-sets of $\mathcal{A}$, namely $\{A : (A, f) \in \mathcal{A}$ for some $f\}$, is an intersecting family. Thus $|\mathcal{A}|$ is at most 2^{r-1} times the maximal size of an intersecting family of r-sets on $[n]$. In particular, if $r \le n/2$ then by the Erdős-Ko-Rado Theorem [1], the maximal size of an intersecting family of r-sets is $\binom{n-1}{r-1}$, and so $|\mathcal{A}| \le 2^{r-1}\binom{n-1}{r-1}$. Thus, the interest of our result is in the case $r > n/2$.

Our method of proof is an averaging method, similar to that used by Katona [2] in his beautiful proof of the Erdős-Ko-Rado Theorem itself. The proof of the inequality itself is rather short, but we have to work a little to prove that the extremal systems are essentially unique.

THEOREM 1. *Let $1 \le r \le n$, and let $\mathcal{A}$ be an intersecting family of signed r-sets on $[n]$. Then $|\mathcal{A}| \le 2^{r-1}\binom{n-1}{r-1}$. If $r < n$, then equality holds if and only if $\mathcal{A} = \{(A, f) \in S_{n,r} : x_0 \in A$, $f(x_0) = \epsilon_0\}$ for some $x_0 \in [n]$ and $\epsilon_0 \in \{-1, 1\}$.*

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PROOF. Call a cyclic ordering of $[n] \times \{-1,1\}$ *good* if $(x,1)$ is opposite $(x,-1)$ for all $x \in [n]$. More formally, a cyclic ordering is a bijection $\sigma : [n] \times \{-1,1\} \to \mathbb{Z}_{2n}$, and σ is said to be good if $\sigma(x,1) = \sigma(x,-1) + n$ for all $x \in [n]$. We say that a signed r-set (A,f) is *compatible* with a cyclic ordering if there is an interval $(x_1,\epsilon_1),\ldots,(x_r,\epsilon_r)$ in the ordering such that for each i we have $x_i \in A$ and $f(x) = \epsilon_i$. In other words, (A,f) is compatible with σ if A can be written as $\{x_1,\ldots,x_r\}$ such that, for some $z \in \mathbb{Z}_{2n}$, we have $\sigma(x_i, f(x_i)) = z + i$ for all i.

Fix a good cyclic ordering σ. How many members of $\mathcal{A}$ are compatible with σ? The corresponding intervals must intersect pairwise. However, since $r \leq (1/2).2n$, it is easy to see that at most r intervals of length r may intersect pairwise. Thus, at most r members of $\mathcal{A}$ are compatible with σ.

Now, there are $n!\,2^n$ good cyclic orderings, and any fixed signed r-set (A,f) is compatible with exactly $2n.r!(n-r)!\,2^{n-r}$ of them. It follows that

$$2n.r!(n-r)!\,2^{n-r}|\mathcal{A}| \leq r.2^n n!,$$

and so $|\mathcal{A}| \leq \binom{n-1}{r-1}2^{r-1}$, proving our inequality.

Let us turn to the case of equality: suppose that $r < n$ and $|\mathcal{A}| = \binom{n-1}{r-1}2^{r-1}$. Then each good cyclic ordering σ is compatible with exactly r members of $\mathcal{A}$. The corresponding intervals intersect pairwise, and so, since $r < (1/2).2n$, they must consist of all intervals containing a fixed element of $[n] \times \{-1,1\}$; we denote this fixed element by $(x^{(\sigma)}, \epsilon^{(\sigma)})$. Note that, in particular, whenever there are two consecutive intervals $(x_1,\epsilon_1),\ldots,(x_r,\epsilon_r)$ and $(x_2,\epsilon_2),\ldots,(x_{r+1},\epsilon_{r+1})$ in the ordering σ, such that the first belongs to $\mathcal{A}$ but the second does not, then the fixed element $(x^{(\sigma)}, \epsilon^{(\sigma)})$ must be precisely (x_1,ϵ_1).

Let τ be the good cyclic ordering given by

$$\tau(x,\epsilon) = \begin{cases} x, & \text{if } \epsilon = 1, \\ x+n, & \text{if } \epsilon = -1, \end{cases}$$

and assume without loss of generality that $(x^{(\tau)}, \epsilon^{(\tau)}) = (n,1)$. Given $1 \leq i < j \leq n-1$, let τ_{ij} be the good cyclic ordering obtained from τ by swapping i and j:

$$\tau_{ij}(x,\epsilon) = \begin{cases} \tau(j,\epsilon), & \text{if } x = i, \\ \tau(i,\epsilon), & \text{if } x = j, \\ \tau(x,\epsilon), & \text{otherwise.} \end{cases}$$

Then, using the remark at the end of the previous paragraph, it is routine to check that we must have $(x^{(\tau_{ij})}, \epsilon^{(\tau_{ij})}) = (x^{(\tau)}, \epsilon^{(\tau)}) = (n,1)$.

Since i and j were arbitrary, it follows that if π is any permutation of $[n-1]$, and we define the cyclic ordering τ_π by

$$\tau_\pi(x,\epsilon) = \begin{cases} \tau(\pi(x),\epsilon), & \text{if } x < n, \\ \tau(x,\epsilon), & \text{if } x = n, \end{cases}$$

then also $(x^{(\tau_\pi)}, \epsilon^{(\tau_\pi)}) = (n,1)$. However, it is clear that if (A,f) is any signed r-set on $[n]$ with $n \in A$ and $f(n) = 1$, then (A,f) corresponds to an interval in the ordering τ_π for some π. Indeed, if $A = \{n, x_1,\ldots,x_s,y_1,\ldots,y_{r-s-1}\}$, with $f(n) = 1$, $f(x_i) = 1$ for all i and $f(y_j) = -1$ for all j, then we may take any permutation π satisfying $\pi(x_1) = n-1, \pi(x_2) = n-2,\ldots,\pi(x_s) = n-s, \pi(y_1) = 1, \pi(y_2) = 2,\ldots,\pi(y_{r-s-1}) = r-s-1$. It follows that $\mathcal{A}$ contains $\{(A,f) \in S_{n,r} : n \in A,\ f(n) = 1\}$, as required. ∎

It is fairly simple to extend Theorem 1 to signed sets which may take more than two values. An *l-signed r-set* is a pair (A,f), where $A \subset [n]$ is an r-set and f is a function from A to $[l]$. We write $S_{n,r,l}$ for the collection of all l-signed r-sets on $[n]$. A family $\mathcal{A}$ of l-signed r-sets is *intersecting* if for any $(A,f),(B,g) \in \mathcal{A}$ there exists $x \in A \cap B$ such that $f(x) = g(x)$.

THEOREM 2. *Let $1 \le r \le n$, and let $\mathcal{A}$ be an intersecting family of l-signed r-sets on $[n]$, where $l \ge 2$. Then $|\mathcal{A}| \le \binom{n-1}{r-1} l^{r-1}$. Unless $l = 2$ and $r = n$, equality holds if and only if $\mathcal{A} = \{(A, f) \in S_{n,r,l} : x_0 \in A,\ f(x_0) = i_0\}$ for some $x_0 \in [n]$ and $i_0 \in [l]$.*

PROOF. Call a cyclic ordering σ of $[n] \times [l]$ (a bijection from $[n] \times [l]$ to $\mathbb{Z}_{nl}$) *good* if for every $x \in [n]$ and $i \in [l]$ there is a $j \in [l]$ with $\sigma(x, i) + n = \sigma(x, j)$. We now proceed just as in the proof of Theorem 1.

Note that, in the case of equality, for $l \ge 3$ we do not need the restriction $r < n$, since if $l \ge 3$ then any $r \le n$ satisfies $r < (1/2).kn$. ∎

We may reformulate Theorem 2 as follows.

THEOREM 2′. Let $l \ge 2$, and let $X = X_1 \cup \ldots \cup X_n$, where the X_i are disjoint sets each of size l. Let $1 \le r \le n$, and let $\mathcal{A}$ be an intersecting family of r-sets on X such that every $A \in \mathcal{A}$ satisfies $|A \cap X_i| \le 1$ for all i. Then $|\mathcal{A}| \le \binom{n-1}{r-1} l^{r-1}$. Moreover, except in the case $l = 2$, $r = n$, equality holds if and only if all members of $\mathcal{A}$ contain a fixed element x_0 of X. ∎

2. AN APPLICATION

Let k be a positive integer, and write $[-k, k]$ for $\{-k, -k+1, \ldots, k\}$. The *grid* on $[-k, k]^n$ is the graph on $[-k, k]^n$ in which $(x_1, \ldots, x_n)$ is joined to $(y_1, \ldots, y_n)$, if for some i we have $|x_i - y_i| = 1$ and $x_j = y_j$ for all $j \ne i$. Given a positive integer d, how large can a subset of $[-k, k]^n$ of diameter d be?

For any $c \in \mathbb{R}^n$ and $r \in \mathbb{R}_+$ let us set

$$B(c, r) = \left\{x \in [-k, k]^n : \sum |x_i - c_i| \le r\right\}.$$

Thus, certainly $B(c, r)$ has diameter at most $2r$. Kleitman and Fellows [3] conjectured that if d is even, then a subset of $[-k, k]^n$ of diameter d has size at most $|B(0, d/2)|$, while if d is odd then the maximum size is $|B((1/2)e_1, d/2)|$ (where $e_1, \ldots, e_n$ denotes the standard basis of $\mathbb{R}^n$). This was proved in [4].

Our aim here is to give a different proof of this result when $d \ge n(k+1)$, in other words for values of d larger than 'half-way'. Not surprisingly, the case d odd is harder than the case d even, and it is there that we shall apply Theorem 1.

We need a small amount of notation. Let G be a subset of $[-k, k]^n$, and for convenience write S_r for $S_{n,r}$. For $(A, f) \in S_r$, we define the *(A,f)-quadrant* of G to be

$$G_{(A,f)} = \left\{x \in [k]^A : \sum_{i \in A} x_i f(i) e_i \in G\right\}.$$

Thus, $|G_{(A,f)}|$ is the number of points $x \in G$ such that $x_i = 0$ for $i \notin A$, and for $i \in A$ the sign of x_i is $f(i)$. In particular,

$$|G| \le 1 + \sum_{r=1}^{n} \sum_{(A,f) \in S_r} \left|G_{(A,f)}\right|,$$

with equality if $0 \in G$.

For $1 \le r \le n$, and any nonnegative integer s, define

$$F_r(s) = \left|\left\{x \in [k]^r : \sum x_i = s\right\}\right|.$$

Thus, $F_r(s) = F_r(r(k+1) - s)$ for all s. It is easy to check that the function F_r is increasing for $s \le r(k+1)/2$, for example, because $[k]^r$ has a symmetric chain decomposition (see [5, Chapter 3]).

We are now ready for our application.

THEOREM 3. *Let G be a subset of $[-k,k]^n$ of diameter d, where $d \geq n(k+1)$. Then*

$$|G| \leq \begin{cases} \left|B\left(0, \frac{d}{2}\right)\right|, & \text{if } d \text{ is even,} \\ \left|B\left(\frac{1}{2}e_1, \frac{d}{2}\right)\right|, & \text{if } d \text{ is odd.} \end{cases}$$

PROOF. Let us start with the case d even. For convenience, write B for $B(0, d/2)$. To show that $|G| \leq |B|$, we shall show that the total number of points in 'opposite quadrants' is at most as large for G as it is for B; in other words, we shall show that for every $1 \leq r \leq n$ and every $(A, f) \in S_r$, we have

$$\left|G_{(A,f)}\right| + \left|G_{(A,-f)}\right| \leq \left|B_{(A,f)}\right| + \left|B_{(A,-f)}\right|.$$

Fix then a signed r-set (A, f). If either of $G_{(A,f)}$, $G_{(A,-f)}$ is empty then certainly $|G_{(A,f)}| + |G_{(A,-f)}| \leq |B_{(A,f)}| + |B_{(A,-f)}|$, since the fact that $d \geq r(k+1)$ implies that $|B_{(A,f)}| + |B_{(A,-f)}| \geq k^n$. So we may assume that $G_{(A,f)}, G_{(A,-f)} \neq \emptyset$.

Choose $x \in G_{(A,f)}$ and $y \in G_{(A,-f)}$ with $\sum x_i$ and $\sum y_i$ maximal: say $\sum x_i = t$ and $\sum y_i = u$. Since G has diameter d, we must have $t + u \leq d$. Now, by the choice of t and u, we have

$$\left|G_{(A,f)}\right| + \left|G_{(A,-f)}\right| \leq \sum_{s=r}^{t} F_r(s) + \sum_{s=r}^{u} F_r(s).$$

Since $t + u \leq d$ and $d \geq r(k+1)$, we have

$$\sum_{s=r}^{t} F_r(s) + \sum_{s=r}^{u} F_r(s) \leq 2\sum_{s=r}^{d/2} F_r(s) = \left|B_{(A,f)}\right| + \left|B_{(A,-f)}\right|.$$

Hence $|G_{(A,f)}| + |G_{(A,-f)}| \leq |B_{(A,f)}| + |B_{(A,-f)}|$, as required.

We now turn to the case d odd. Write B for $B((1/2)e_1, d/2)$. For any $1 \leq r \leq n$ and any $(A, f) \in S_r$, it follows just as above that

$$\left|G_{(A,f)}\right| + \left|G_{(A,-f)}\right| \leq \sum_{s=r}^{(d+1)/2} F_r(s) + \sum_{s=r}^{(d-1)/2} F_r(s).$$

Also, it is certainly true that if for all $x \in G_{(A,f)}$ and all $x \in G_{(A,-f)}$, we have $\sum x_i \leq (d-1)/2$ then

$$\left|G_{(A,f)}\right| + \left|G_{(A,-f)}\right| \leq 2\sum_{s=r}^{(d-1)/2} F_r(s).$$

Now, $B_{(A,f)} = \{x \in [k]^A : \sum x_i \leq (d+1)/2\}$ if $1 \in A$ and $f(1) = 1$, and $\{x \in [k]^A : \sum x_i \leq (d-1)/2\}$ otherwise. Thus, to complete the proof that $|G| \leq |B|$, it will suffice to show that, for each $1 \leq r \leq n$, the number of signed r-sets (A, f) such that $\sum x_i \geq (d+1)/2$ for some $x \in G_{(A,f)}$ is at most $2^{r-1}\binom{n-1}{r-1}$.

Let then

$$\mathcal{A} = \left\{(A, f) \in S_r : \sum x_i \geq \frac{(d+1)}{2} \text{ for some } x \in G_{(A,f)}\right\}.$$

We claim that $\mathcal{A}$ is an intersecting family. Indeed, suppose that, to the contrary, there are signed r-sets $(A, f), (C, g) \in \mathcal{A}$ such that for all $i \in A \cap C$ we have $f(i) \neq g(i)$. Then, for any $x \in G_{(A,f)}$ and $y \in G_{(C,g)}$, the distance between $\sum x_i f(i)e_i$ and $\sum y_i g(i)e_i$ is $\sum x_i + \sum y_i$. However, this implies that $\mathcal{A}$ has diameter at least $d+1$.

Thus, $\mathcal{A}$ is an intersecting family of signed r-sets on $[n]$, and so by Theorem 1 we have $|\mathcal{A}| \leq 2^{r-1}\binom{n-1}{r-1}$, as required. ∎

We remark that it is not hard to see that if equality holds in Theorem 3, and d is even, then in fact $G = B(0, d/2)$. Also, using the case of equality in Theorem 1, one may check that if equality holds in Theorem 3 and d is odd then $G = B(\{-1, 1\}(1/2)e_i, (d/2))$ for some i and choice of sign.

As we mentioned above, it was proved in [4] that the conclusion of Theorem 3 remains valid without the restriction that $d \geq n(k+1)$. The same paper also contains several related results.

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